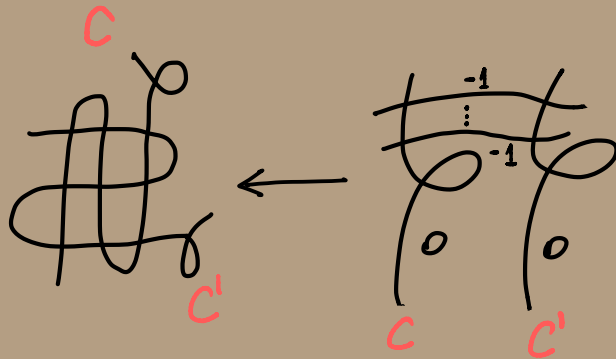




«Mild degenerations of Elliptic surfaces»



CLA 2026
LIMA 14 agosto
10:00 - 10:50
Giancarlo Urzúa

"Diferencias entre ser canónicamente negativo o positivo"

Coloquio Peruano at PUCP, 12-Agosto-25

- +
- << Classification of QHD Horikawa surfaces >> M. Conedo, U. 2025
 - << Classification of Horikawa surfaces with T-singularities >> V. Mouruel, J. Negrete, U. 2024 (+ parte II 2025-26)
 - << Tropical methods for stable octic surfaces >> 2024 J. Evans, A. Simonetti, U.
 - << Exotic surfaces >> J. Reyes, U. 2022
 - << Savage surfaces >> S. Truncoso, U. 2019
 - << Chern slopes of simply connected complex surfaces of general type are dense in $[2, 3]$ >> X. Roulleau, U. 2014

$K > 0$
General
type

○ : << Rational homology disk degenerations of elliptic surfaces >> M. Conedo, U. 2026

$K = 0$
CALABI-YAU

-
- << Wahl singularities on degenerations of del Pezzo surfaces >> U., J.P. Zúñiga. 2025
 - << The birational geometry of Mordoo numbers >> U. J.P. Zúñiga. 2023

$K < 0$
FANO

Elliptic curve = nonsingular projective curve $|_{\mathbb{C}}$ of genus 1.



$$\simeq \mathbb{C}/\langle 1, \tau \rangle \simeq \{zy^2 = x(x-z)(x-\lambda z)\} \subset \mathbb{P}_{\mathbb{C}}^2$$

$\text{Im}(\tau) > 0 \qquad \lambda \neq 0, 1$



K. Kodaira (1915-1997) studied one parameter degenerations of elliptic curves by 1963.

Kodaira's result says:

$C \subset S = \text{non-singular surface} \mid \mathbb{C}$ with $C_t = \text{elliptic curve for } t \neq 0$
 $\downarrow \quad \downarrow f = \text{fibration}$
 $0 \in \mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$
 $C = C_0$ with no $\mathbb{P}_{\mathbb{C}}^1$ such that $\mathbb{P}_{\mathbb{C}}^1 \cdot \mathbb{P}_{\mathbb{C}}^1 = -1$ ((-1)-curve)

$\Rightarrow C$ is one of the following:

$m I_0 = \text{elliptic curve}$

$m I_1 = \wp$ nodal rational

$m I_n = \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ \diagdown \quad \diagup \end{array} \quad n \mathbb{P}_{\mathbb{C}}^1 \text{'s in a nodal curve}$

$II = \prec$ cuspidal rational

$III = \chi$ 2 $\mathbb{P}_{\mathbb{C}}^1$'s & tacnode

$IV = \times$ 3 $\mathbb{P}_{\mathbb{C}}^1$'s & triple point

$I_0^* = \begin{array}{c} | \\ | \\ | \\ | \end{array}$

$I_n^* = \underbrace{\begin{array}{c} \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array}}_{n+1}$

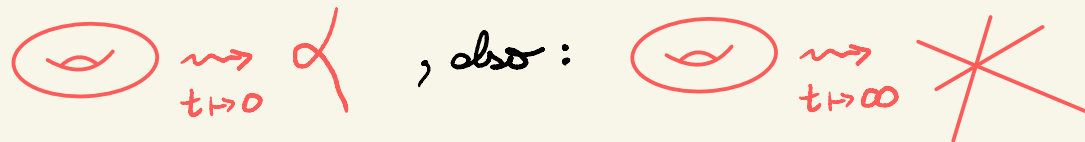
$II^* = \begin{array}{c} \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array}$

$III^* = \begin{array}{c} \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array}$

$IV^* = \begin{array}{c} \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array}$

Example : just take the pencil

$$C_t = \{ y^2 z - x^2(x-z) + t(y-z)(y-2z)(y-3z) = 0 \} \subset \mathbb{P}_{\mathbb{C}}^2$$



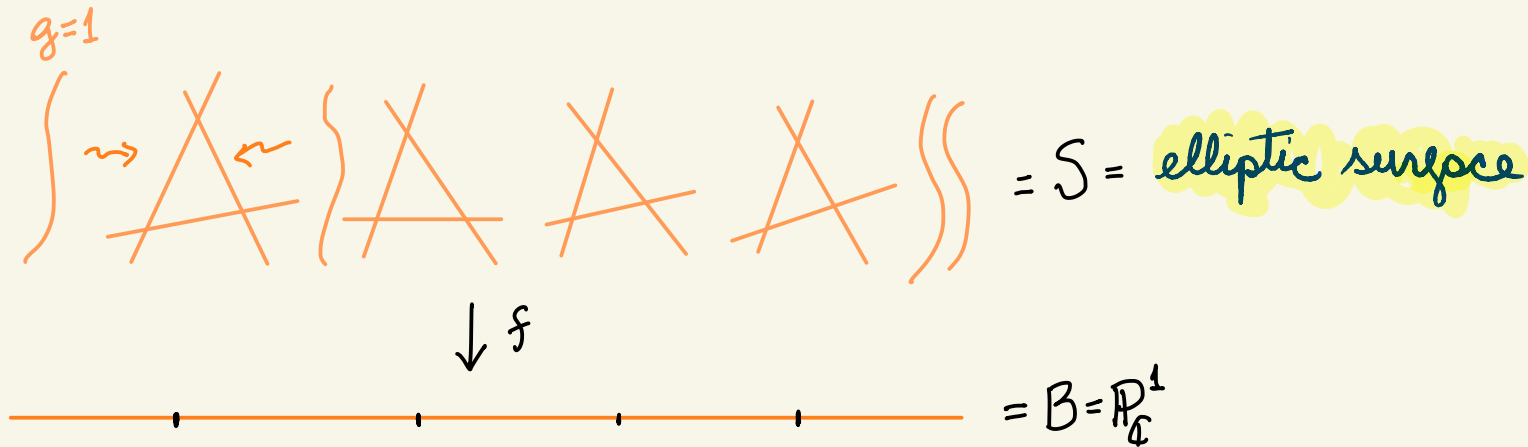
To be as in definition we need to Blow-up 9 times over $\mathbb{P}_{\mathbb{C}}^2$

$$\Rightarrow \mathbb{P}_{\mathbb{C}}^2 \longleftarrow S = \text{an elliptic fibration.}$$

$$\downarrow$$

$$\mathbb{P}_{\mathbb{C}}'$$

Definition : An elliptic surface is a nonsingular projective surface S with an elliptic fibration $S \xrightarrow{f} B = \text{nonsingular projective curve}$, i.e., up to finite points in B the fiber is an elliptic curve .



obs1: If singular fiber has a (-1) -curve $\Rightarrow$ contract it to a new elliptic surface. At the end, one has the Kodaira fibers & relatively minimal elliptic surface.

obs2: Classification (κ = Kodaira dimension)

$\kappa = -\infty$: They are rational and come from pencils of plane curves $\longrightarrow$ Cubics: have sections.

$\longrightarrow$ others: no sections and one multiple fiber.
or trivial elliptic $C \times \mathbb{P}_C^1 \longrightarrow \mathbb{P}_C^1$.

$\kappa = 0$: various examples in every class ($K3$, Enriques, etc).

$\kappa = 1$: The most interesting **PROPER ELLIPTIC SURFACES**.

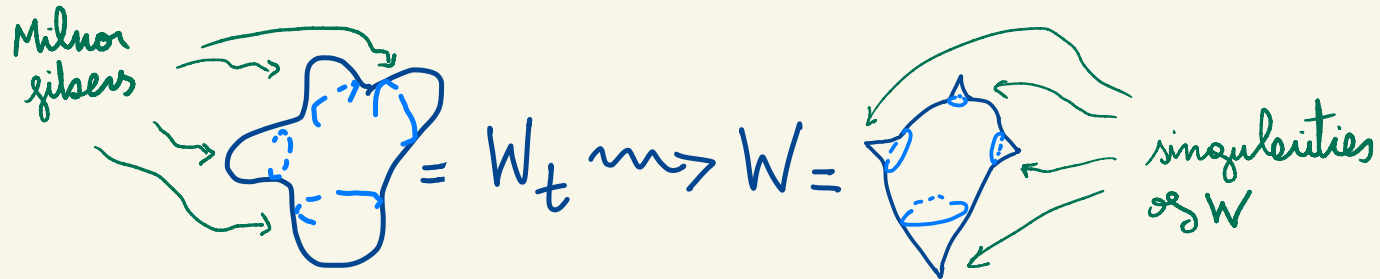
$\kappa = 2$: There are none (κ^2 must be zero!).

we just saw the degenerations of elliptic curves.

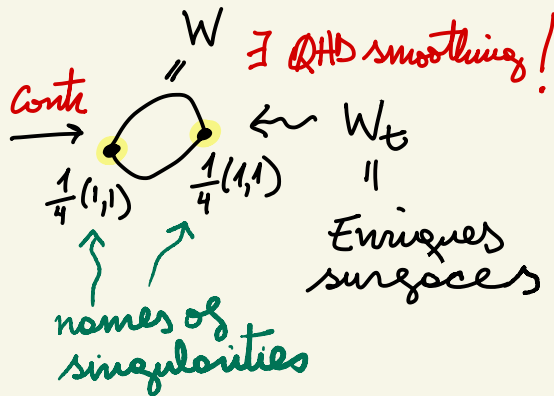
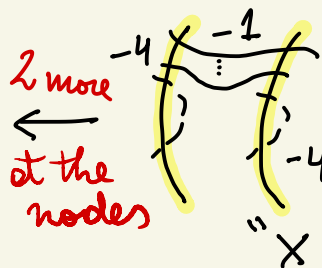
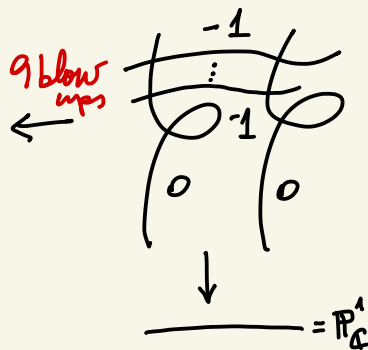
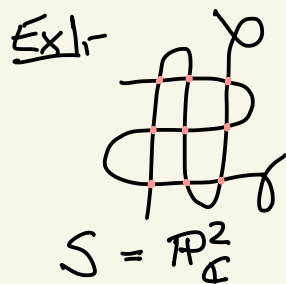
What are the degenerations of elliptic surfaces?

we mean: normal proj. surface $W \subset \mathcal{W} = \text{normal 3-fold}$
 $\downarrow \quad \downarrow$ proper holomorphic map
 $0 \in \mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$

so that $W_t = \text{elliptic surface}$ for $t \neq 0$.



Constructions of degenerations tend to be abstract ...



Let us first think about degenerations with only

ADE = Rational double points = Kleinian = simple = Du Val =
" Rational Gorenstein
" Canonical



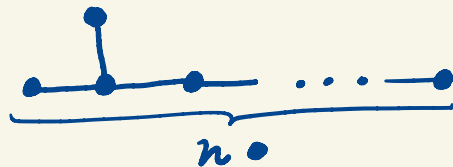
(1903 - 1987)

singularities.

All equations of Du Val singularities in $\mathbb{C}^3$:



$$A_n \ (n \geq 1) : z^2 + x^2 + y^{n+1} = 0$$



$$D_n \ (n \geq 4) : z^2 + y(x^2 + y^{n-2}) = 0$$

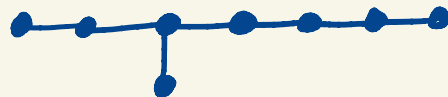
$$E_6 : z^2 + x^3 + y^4 = 0$$



$$E_7 : z^2 + x(x^2 + y^3) = 0$$



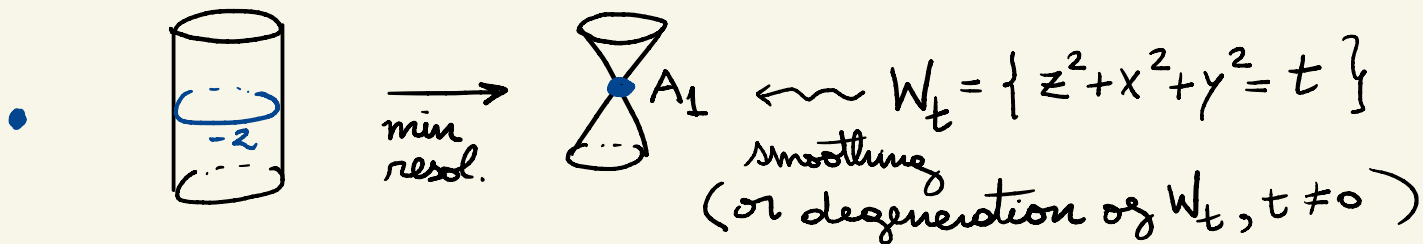
$$E_8 : z^2 + x^3 + y^5 = 0$$



In blue the dual graphs of the minimal resolutions.

$$\bullet^2 = -2$$

Example :

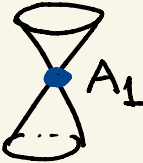


Theorem (E. Brieskorn 1970 - G.N. Tian 1969) Every degeneration of surfaces with only ADE singularities can be simultaneously resolved (up to a base change).

This means : we can think about it as a smooth deformation, there are no singularities involved.

That was too mild ...

- For a surface (normal) singularity $p \in \bar{W} \subset \bar{X}$ smoothing,
the Milnor fiber is $\bar{W}_t$, $t \neq 0$.
 $\downarrow$
 $\mathbb{D}$
- It is a 4-manifold with boundary $\simeq$ link of $p \in \bar{W}$.
- It is homotopically equivalent to a CW complex of dim 2
and $b_0 = 1$, $b_1 = 0$ and $b_2 = \text{Milnor \# of the smoothing} = \mu \geq 0$
(Steinbrick)

Example:  $\leftarrow W_t = \{z^2 + x^2 + y^2 = t\}$ $\Rightarrow \mu = 1$
smoothing

There is one vanishing cycle in $H_2(W_t)$: The sphere $z^2 + x^2 + y^2 = t$
 $t \in \mathbb{R}_{>0}$

$\mu(A_n) = n$, $\mu(D_n) = n$, $\mu(E_n) = n$ (one smoothing)

Declaration : The mildest degenerations happen when we induce a $\mu=0$ smoothing at every singularity in W .

If $W_t \rightsquigarrow W$ is such $\Rightarrow H_2(W_t) \subset H_2(W)$ with finite index ; $K_{W_t}^2$, $\chi_{\text{top}}(W_t)$, $g(W_t)$, $pg(W_t)$ are constant.

Note that ADE singularities are out !

Let us call them QHD singularities (admit a rational homology disk smoothing) : which are they ?!

(No vanishing cycles here !)

In the 80s, J. Wohl found many, among them:



$$\mathbb{C}^2 \xrightarrow{T} \mathbb{C}^2$$

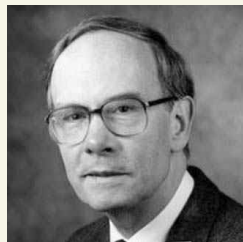
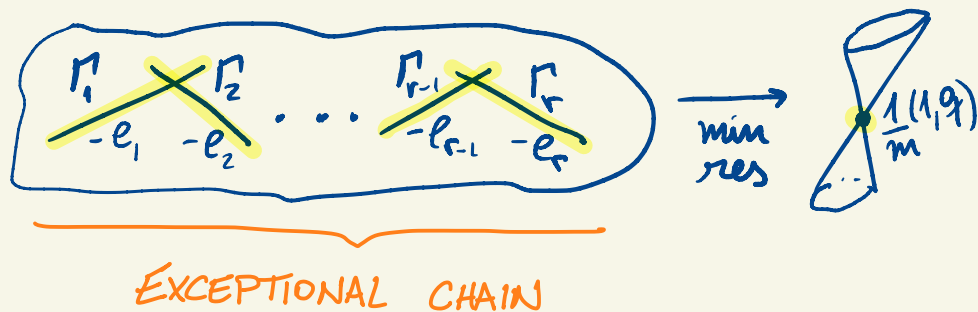
$$(x, y) \mapsto (zx, z^q y)$$

$0 < q < m$ coprime
and z m^{th} primitive
root of 1

$$\Rightarrow \frac{1}{m}(1, q) \in \bigvee = \mathbb{C}^2 / \langle T \rangle \text{ is } \underline{\text{cyclic quot. sing.}}$$

In this set of singularities: $\mu=0 \Leftrightarrow m=n^2$ with $\gcd(n, a)=1$
 $q=na-1$
Wohl singularities

In general, minimal resolution:



Hinzbruch
(1927-2012)



Saito
(1876-1953)

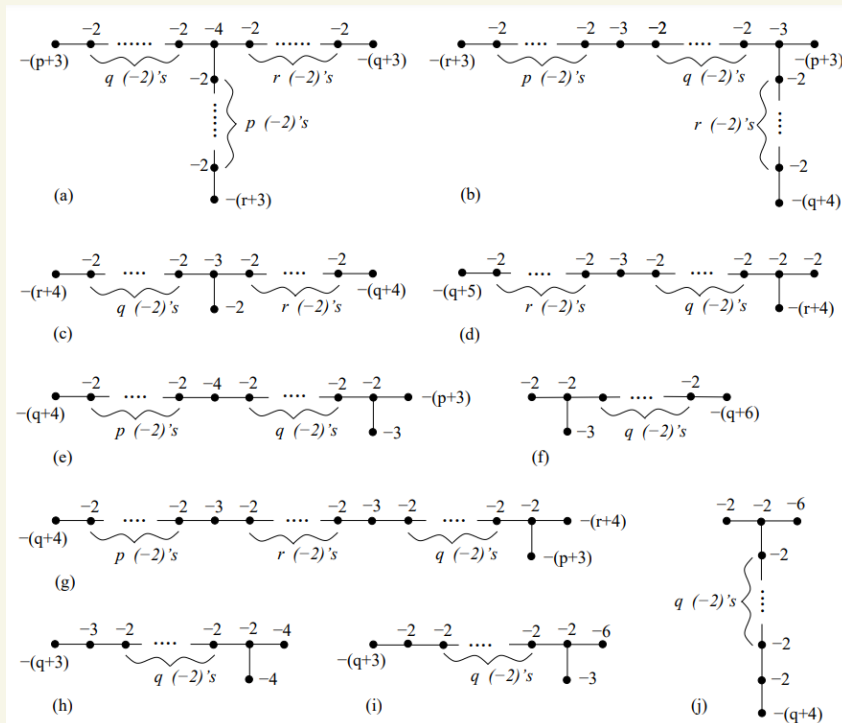
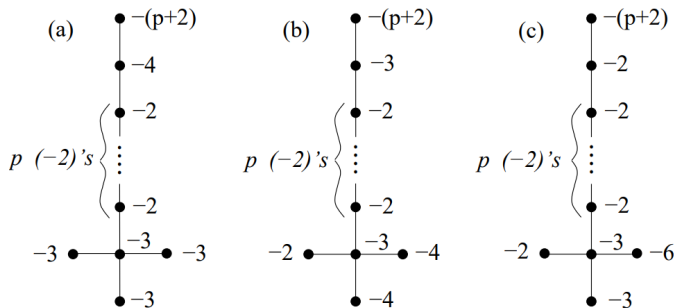
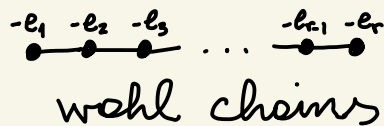
continued fraction

$$\Gamma_i^2 = -e_i \leq -2 \quad \forall i$$

$$\Gamma_i \simeq \mathbb{P}^1$$

$$\frac{m}{g} = e_1 - \frac{1}{e_2 - \frac{1}{\dots - \frac{1}{e_r}}}$$

List of (minimal resolutions of) known QHD : (due to Wahl, Stipsicz-Szabó-Wahl, Buhol-Stipsicz)



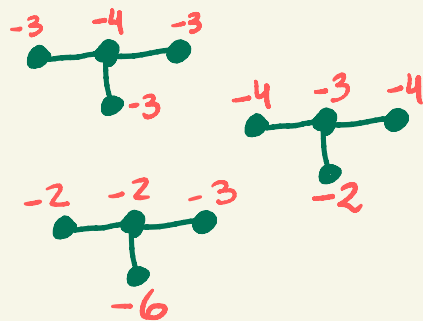
Wahl Conjecture:
This is the complete list
of families of QHD.

According to birational geometry, we should pay attention to (log) terminal and (log) canonical singularities.

In dimension 2

log Canonical	log terminal	Canonical	Terminal
there is a list!	$\parallel$ quotient singularities	$\parallel$ Du Val	$\emptyset$

For QAD singularities



wahl singularities

$$\frac{1}{n^2} (1, na-1)$$

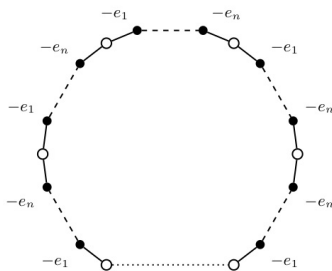
Thus almost all of them are not in the usual list!



Y. Kawamote proves in "Moderate degenerations of algebraic surfaces" (1992) :

Theorem 1.1. *Let W be a $\mathbb{Q}$ HD surface with only Wahl singularities and K_W nef. Assume that $(W \subset \mathcal{W}) \rightarrow (0 \in \mathbb{D})$ is a $\mathbb{Q}$ HD smoothing with Kodaira dimension $\kappa(W_t) = 1$ for $t \neq 0$. Then W admits an elliptic fibration $W \rightarrow B$ over some curve B , together with a classification of the fibers containing singularities of W .*

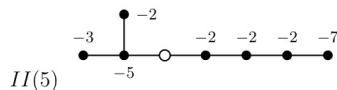
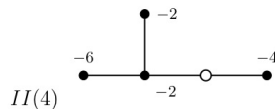
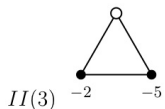
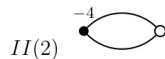
Type I_d



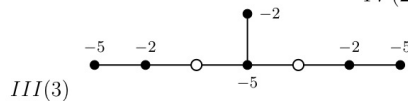
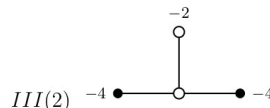
$yI_d(n, a)$

We have d Wahl chains $[e_1, \dots, e_r]$ over an I_d fiber of multiplicity y .

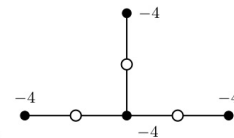
Type II



Type III



Type IV



Theorem (M. Conedo, U. 2025): Let W be a QAD degeneration of proper elliptic surfaces. Then W admits an elliptic fibration and the dual graphs of minimal resolutions of elliptic fibers:

I

VALENCY 3 (NON LOG CANONICAL)								
Type II								
II.V3.a	II.V3.b	II.V3.c	II.V3.d	II.V3.e	II.V3.f	II.V3.g	II.V3.i	II.V3.j
$\begin{cases} p=0 \\ q=0 \\ r=3 \end{cases}$	$\begin{cases} p=0 \\ q=2 \\ r=0 \end{cases}$	$\begin{cases} q=0 \\ r=2 \end{cases}$	$r=2$	$p=3$	—	$\begin{cases} p=0 \\ r=2 \end{cases}$	—	—
Type III								
III.V3.a	III.V3.b	III.V3.c	III.V3.d	III.V3.g	III.V3.h	IV.V3.a	IV.V3.b	IV.V3.e
$\begin{cases} p=1 \\ q=0 \\ r=2 \end{cases}$	$\begin{cases} q=0 \\ r=0 \end{cases}$	$p=q$	$r=0$	$\begin{cases} p=1 \\ r=0 \end{cases}$	—	$p=q=r$	$\begin{cases} p=0 \\ r=q+1 \end{cases}$	$p=0$

+ sliding degenerations

+ Kawamata's theorem.

VALENCY 3 (LOG CANONICAL)		
Type III	Type III	Type IV
II.V3.f.0	III.V3.c.0	IV.V3.a.0
$q=0$	$p=q=0$	$p=q=r=0$

VALENCY 4		
Type II	Type III	Type IV
II.V4.c	III.V4.b	IV.V4.a

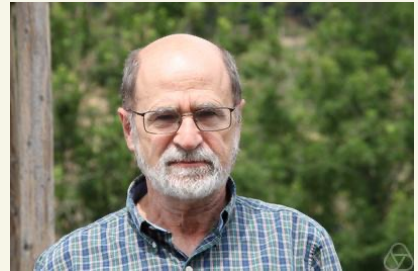
Why did we want to do this ?

- (1) Part of the big goal classification of all QHD degenerations (e.g. general type/Hirakawa)
- (2) A recent work 2025 of D. Lee and Y. Lee giving new degenerations of Dolgachev surfaces.

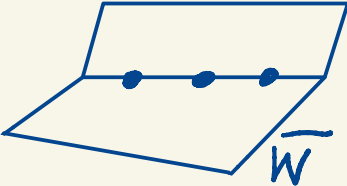
Definition: A Dolgachev surface $D_{A,B}$ is a minimal elliptic surface $D_{A,B} \rightarrow \mathbb{P}^1$ with $p_g(D_{A,B}) = 0$ and exactly 2 multiple fibers with multiplicities A and B .

we have $\pi_1(D_{A,B}) \simeq \mathbb{Z}/\gcd(A,B)$.

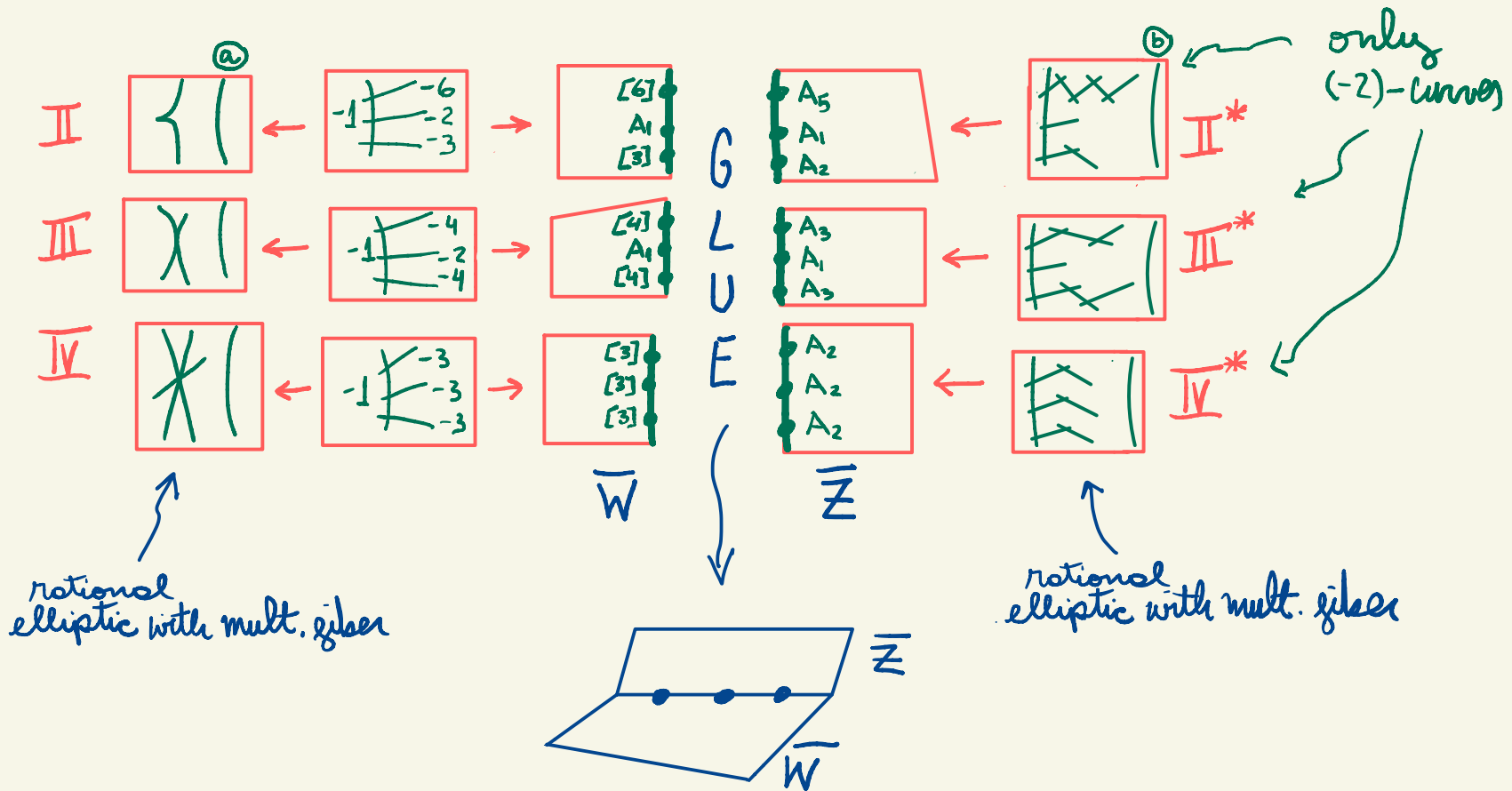
$D_{2,2}$ = Enriques surface.



Lee-Lee examples : Consider $D_{p,q}$ Dolgachev surfaces with p, q coprime (ie $\pi_1 = 1$).

$\Rightarrow D_{p,q} \rightsquigarrow$  $\bar{Z}$, where $\bar{W} \cap \bar{Z} = P_G^1 =: \Delta$ and

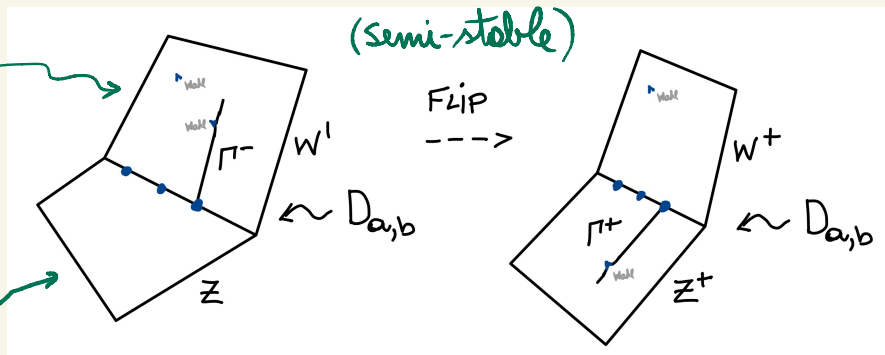
the singular projective surfaces $\bar{W}$ and $\bar{Z}$ are:



Marcos and I perform this birational geometry :

Proper
transformation
of W

Compactly
fibred
Milnor fibres



we prove that $Z^+ W^+$
has no local-to-global
obstructions to deform



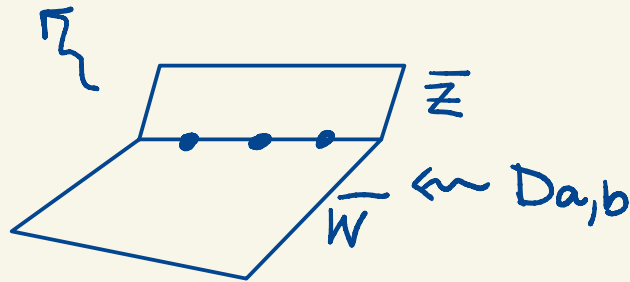
smooth wehl sing.

Seibert blow-up (after J. wehl)



$W \leftarrow D_{a,b}$ = Dolgachev surface
for some a, b
not nec. coprime

our initial degeneration
with QHD singularities



This is Lee-Lee degeneration
(Canonical class is neg)

We have the following theorems with Marcos Conedo.

II Theorem: Let $a, b \geq 2$ be integers with $ab > 4$. Then for each QHD singularity in our classification where denominator of λ is a ,

M-mod	λ	M-mod	λ	M-mod	λ	M-mod	λ
II.v3.a	$6/7$	II.v3.i	$\frac{8+3q}{9+3q}$	III.v3.g	$\frac{6+2q}{7+2q}$	$yI_d(n, a)$	$\frac{n-1}{yn}$
II.v3.b	$7/8$	II.v3.j	$\frac{7+2q}{8+2q}$	III.v3.h	$\frac{5+2q}{6+2q}$	II(2)	$1/2$
II.v3.c	$6/7$	II.v4.c	$\frac{11+6p}{12+6p}$	III.v4.b	$\frac{7+4p}{8+4p}$	II(3)	$2/3$
II.v3.d	$\frac{8+2q}{9+2q}$	III.v3.a	$4/5$	IV.v3.a	$\frac{2+p}{3+p}$	II(4)	$3/4$
II.v3.e	$\frac{9+3q}{10+3q}$	III.v3.b	$4/5$	IV.v3.b	$\frac{3+q}{4+q}$	II(5)	$4/5$
II.v3.f.0	$5/6$	III.v3.c.0	$3/4$	IV.v3.c	$\frac{3+q}{4+q}$	III(2)	$1/2$
II.v3.f	$\frac{5+q}{6+q}$	III.v3.c	$\frac{3+p}{4+p}$	IV.v4.a.0	$5/6$	III(3)	$2/3$
II.v3.g	$\frac{10+3q}{11+3q}$	III.v3.d	$\frac{4+q}{5+q}$	IV.v4.a	$\frac{5+3p}{6+3p}$	IV(2)	$1/2$

we have a QHD degeneration of surfaces in $D_{a,b}$ to a W with only that singularity.

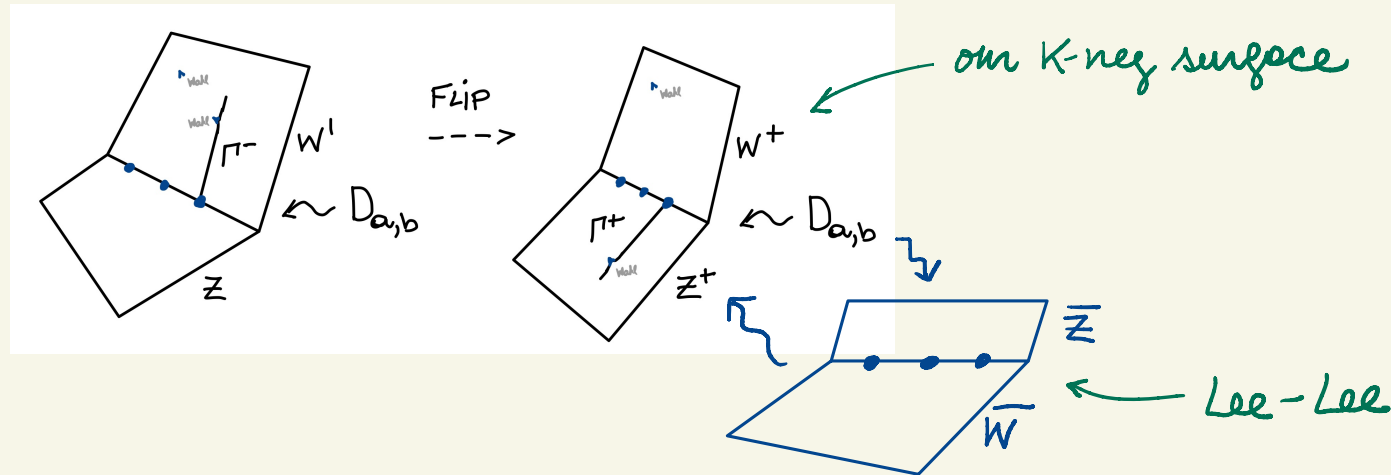
we classify (and realize) every such degeneration of Dolgachev surfaces } "QHD boundary divisors"

In relation to Lee-Lee examples, we prove the following.

III

Theorem: The broken surface $W^+ \cup Z^+$ has a nonsingular 10-dimensional $\mathbb{Q}$ -Gorenstein deformation space, and the surfaces preserving $\Delta = W^+ \cap Z^+$ form a divisor and are Lee-Lee examples.

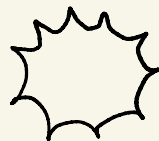
(there was no direct way to see this from Lee-Lee.)



What about QHD degenerations for other surfaces?

Some ending open questions and progress:

- (1) If $W_t = \mathbb{P}_C^2 \Rightarrow$ full classification (with M. Conedo).
- (2) If W_t del Pezzo $\Rightarrow$ full classification when singularities are Wahl (with J.P. Zúñiga). Rest QHD is open.
- (3) If W_t rational and $K_{W_t}^2 \leq 0 \Rightarrow$ open. (S. Flores $K_{W_t}^2 = 0$ and Wahl sing.)
- (4) If $K(W_t) = 0$ and W singular $\Rightarrow W_t =$ Enriques surfaces and we know all degenerations (including all QHD?)



$= W \leftarrow$ Enriques surfaces (I have a paper about it!)

10 sing = Maximal degeneration, all $\frac{1}{4}(1,1)$ singularity

COBLE-MUKAI
Lattice

(5) $\exists_g K(W_t) = 1 \Rightarrow$ This talk was about that!

(6) $\exists_g K(W_t) = 2 \Leftrightarrow W_t$ is of general type $\Rightarrow$ Many open questions.

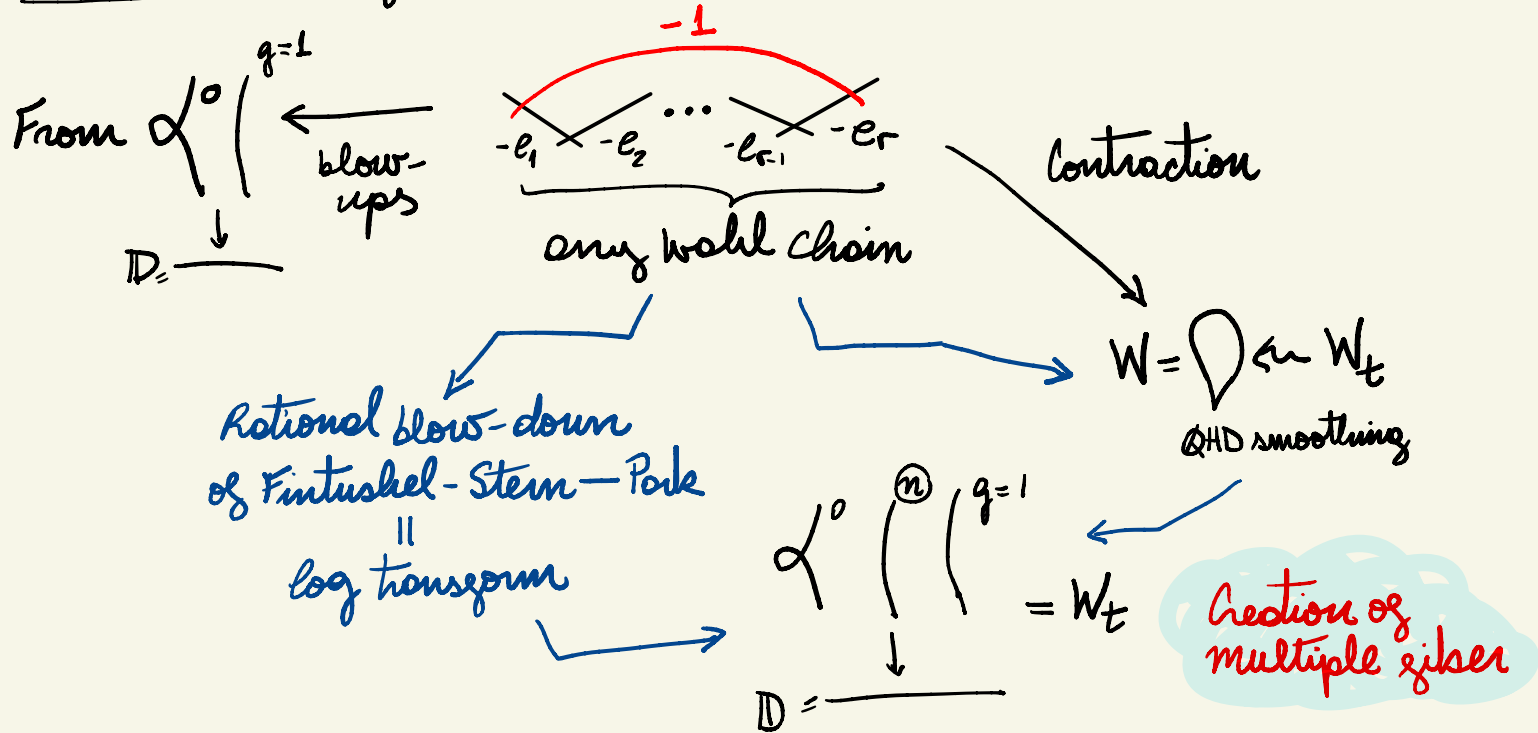
- When $K^2 = 2p_g - 4$ or $2p_g - 3$, we have classification of all such W with only Wahl singularities.

[V. Monreal - J. Noguera - U 2024 $K^2 = 2p_g - 4$] [Same authors 2027? $K^2 = 2p_g - 3$]

For QHD and $K^2 = 2p_g - 4$: work in progress with M. Conedera.

- Many examples for $K^2 > 2p_g - 3$ but no classification.
- $p_g = 0$ surfaces are very interesting: Thousands of degenerations using computer searches but no coarse classification (with J. Reyes).

observation: logarithmic transformation.



For QHD general singularities we create also multiple fibers.